PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Phil. Trans. R. Soc. Lond. A 1980 **296**, 129-148
doi: 10.1098/rsta.1980.0160

Phil. Trans. R. Soc. Lond. A **296**, 129–148 (1980) [129]
Printed in Great Britain

Towards a Lagrangian-mean description of stratospheric circulations and chemical transports

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The theory of Lagrangian means and Stokes drifts is reviewed, with particular attention to recent developments and to possibilities for applying the ideas to the stratosphere. The properties of Lagrangian means for finite-amplitude disturbances in spherical geometry pose certain conceptual and technical problems, which appear unavoidable if the notion of Lagrangian mean is to be defined in an exact, coordinate-independent manner compatible with Stokes's classical theory.

1. Motivation

A number of stratospheric phenomena, which seem obscure and complicated within the standard description using Eulerian zonal means and deviations, are better appreciated from a Lagrangian viewpoint (see, for example, Reed 1955; Staley 1960; Danielsen 1968; Mahlman 1969; Shapiro 1978; Dunkerton 1978; Wallace 1978; Matsuno 1979). One example is the well known propensity of Eulerian eddy fluxes of quasi-conservative quantities to have almost any direction relative to their local mean gradients (see, for example, Mahlman 1975, p. 138; Matsuno 1979; Plumb 1979). The mean behaviour of the same quasi-conservative quantities, e.g. the mean motion of certain trace gases, comprises another, related, example (see, for example, Brewer 1949; Dobson 1956; references cited above).

A fictitious stratosphere in which potential temperature θ and tracers are exactly conservative, i.e.

$$\mathrm{D}X/\mathrm{D}t = 0, \tag{1}$$

where $\mathrm{D}/\mathrm{D}t$ is the rate of change following an air parcel, and X stands for θ or for tracer mixing ratio, presents an impenetrable barrier to tracers if statically stable everywhere. Each isentropic surface is a material surface, and no tracer satisfying (1) can be carried across it. This statement is obvious, but the fact that it holds whether or not planetary waves or other disturbances to zonal symmetry are present is significant. It reminds us that the Eulerian eddy fluxes of tracers associated with such disturbances cannot give rise to any net tracer transport across isentropic surfaces, except to the extent that the right-hand side of (1) is not zero. If our 'stratosphere' is frictionless, as well as satisfying (1) with $X = \theta$, then surfaces of constant potential vorticity are similarly impenetrable.

The Eulerian zonal mean of equation (1) is

$$\left(\frac{\partial}{\partial t}+\bar{v}\,\frac{\partial}{\partial y}+\bar{\omega}\,\frac{\partial}{\partial p}\right)\bar{X} = -\frac{1}{\cos\varphi}\,\frac{\partial}{\partial y}\,(\overline{v'X'}\cos\varphi)-\frac{\partial}{\partial p}\,(\overline{\omega'X'}), \tag{2}$$

† This paper is JISAO contribution no. 4.

where overbars and primes denote Eulerian zonal means and departures therefrom, v and ω are the usual measures of meridional and vertical velocity in a coordinate system in which the vertical 'coordinate' is pressure p, and y is northward distance, equal to latitude φ times the Earth's radius. If the mean state of our fictitious 'stratosphere' undergoes no systematic changes as time goes on, there must be a long term tendency for the effects of the eddy-flux terms on the right of equation (2) to be cancelled by the effects of the Eulerian-mean meridional circulation $(\bar{v}, \bar{\omega})$ on the left (see, for example, Sawyer 1965, p. 414). As is well known, such cancellations are also observed in the real stratosphere, where for many of the quantities X of interest the right-hand side of (1) is often small even if not zero. These considerations, then, all serve to remind us that the eddy fluxes arising from taking Eulerian means are not very directly related to tracer transport; and they suggest that some kind of Lagrangian averaging (see, for example, Stokes 1847; Riehl & Fultz 1957; Krishnamurti 1961; Mahlman 1969; Kida 1977; Dunkerton 1978; Matsuno & Nakamura 1979; Matsuno 1979) might be a more natural way of defining the mean state of the stratosphere, for use in two dimensional descriptions or models of tracer behaviour.

The simplest concept of Lagrangian averaging is Stokes's classical idea of taking the time mean following a single air parcel (for a good discussion of the classical theory see Longuet-Higgins 1969). This idea is sometimes all that is needed in practice; but it has inherent limitations. If we wish to speak of the Lagrangian-mean velocity at a given point in space, the classical concept cannot apply in any exact sense, because the air parcel we are following will generally wander away from the point under consideration. Thus any exact theory of the Lagrangian-mean velocity *field* must abandon the simple concept of the mean following a single air parcel (Andrews & McIntyre 1978*b* and references, hereafter A.M.). The result will be neither a pure Lagrangian nor a pure Eulerian description but, rather, a *hybrid* Eulerian–Lagrangian description of the motion.

The result of requiring that the concept of 'Lagrangian–mean velocity field' be defined exactly, and that it reduce to the classical theory when the approximations inherent in the latter are applicable, is the generalized Lagrangian-mean (g.l.m.) description of wave, mean-flow interaction developed independently by A.M. and by Bretherton (1979). That theory has provided a powerful analytical key to problems concerning the interaction of waves and mean flows (see, for example, Andrews & McIntyre 1976, 1978*a–c*; Bretherton 1979; Dunkerton 1979; Grimshaw 1979; Leibovich 1979; McIntyre 1979). One reason is that the theory avoids the indirectness of equation (2): when (1) holds the equation for the generalized Lagrangian mean $\bar{X}^{\mathrm{L}}$ of X (to be defined below) can be shown to take the simple form

$$\left(\frac{\partial}{\partial t}+\bar{v}^{\mathrm{L}}\frac{\partial}{\partial y}+\bar{\omega}^{\mathrm{L}}\frac{\partial}{\partial p}\right)\bar{X}^{\mathrm{L}} = 0 \tag{3}$$

(A.M., equation 2.22). That is, it has the same form as (2) except that there are *no eddy-flux terms* on the right. This is not to say that the generalized Lagrangian-mean theory holds all eddy effects to be unimportant; it merely highlights the fact, already mentioned, that the effects of wave or eddy motion on net heat and tracer transport are necessarily bound up with departures from conservative motion, and not necessarily with wave activity *per se*. (Mahlman *et al.* (1979) refer to such statements as 'non-transport theorems'.) Departures from conservative motion entail the appearance of some quantity $Q_X \neq 0$ on the right-hand side of (1), representing effects of small-scale turbulence as well as radiation and photochemistry; the corresponding term then appearing on the right of (3) is $\bar{Q}^{\mathrm{L}}_{X}$. The value of $\bar{Q}^{\mathrm{L}}_{X}$ can be affected by eddy motions (just as can its Eulerian-mean counterpart $\bar{Q}$); such motions, therefore, are certainly important in the equation for $\bar{X}^{\mathrm{L}}$, but they are important *only* in so far as they promote mixing or other departures from conservative motion.

In view of these remarks it is natural to hope that the g.l.m. theory could be applied straightforwardly to the stratosphere, or to numerical models thereof. There are, however, a number of difficulties in the way of using the theory as a practical diagnostic tool, and it is one purpose of this paper to discuss, or at least to state explicitly, what those difficulties seem to be at our present stage of thinking. This is done in § 4 below, after reviewing the basic analytical theory, and its relation with classical concepts, in § 2. In § 3 the important problem of 're-initializing' the theory is discussed briefly; this, too, can have its difficulties.

If blind application of Lagrangian-mean theory is impracticable, this may of course be a good thing in the long run. It can be argued that the very ease with which Eulerian-mean quantities can be calculated, for any complicated fluid flow, has made them something of a two-edged weapon. Elucidation of rates of conversion between mean and eddy energies, for instance, has sometimes been mistaken for a full understanding of the eddy dynamics involved. What the g.l.m. theory does provide at its present stage of development is a precise analytical structure which, first of all, makes clear what is involved in generalizing the classical notions of Lagrangian mean and Stokes drift, and which, secondly, makes available certain analytical tools whose use might prove helpful in the search for a fundamentally improved understanding of the stratosphere. Just how those tools can best be used is not likely to be clear until some detailed applications have been attempted; and it is hoped that the present discussion will help stimulate researchers to tackle some of the intriguing problems involved.

2. Basic theoretical ideas

2.1. *Classical and generalized Lagrangian means*

We start by expressing the classical notion of Lagrangian mean in a form suitable for subsequent generalization, following A.M. For the purposes of exposition we shall forget about pressure coordinates for the present, and use ordinary Cartesian coordinates.† The essential idea is to average over positions displaced by the disturbance motion. In other words, if $\phi(\boldsymbol{x}, t)$ is the quantity to be averaged and $\overline{\phi(\boldsymbol{x}, t)}$ its Eulerian mean at $\boldsymbol{x}$ in any of the usual senses (time, space, ensemble, etc.), then the corresponding Lagrangian mean is defined as

$$\bar{\phi}^{\mathrm{L}}(\boldsymbol{x}, t) \equiv \overline{\phi(\boldsymbol{x}, t)}^{\mathrm{L}} \equiv \overline{\phi\{\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x}, t), t\}}, \tag{4}$$

where $\boldsymbol{\xi}(\boldsymbol{x}, t)$ is the disturbance-induced *displacement* of the fluid parcel, in a sense to be elucidated. (If the disturbance motion is the dominant motion, the meaning of 'displacement' is fairly obvious *ab initio* and is such that if $\overline{(\;)}$ is the time mean at a fixed point, then $\overline{(\;)}^{\mathrm{L}}$ is the time mean following a single fluid parcel, at least approximately.) The Stokes correction to $\bar{\phi}$ is defined as the difference between the Eulerian and the Lagrangian mean:

$$\overline{\phi(\boldsymbol{x}, t)}^{\mathrm{S}} \equiv \overline{\phi(\boldsymbol{x}, t)}^{\mathrm{L}} - \overline{\phi(\boldsymbol{x}, t)}. \tag{5}$$

† This involves no loss of generality, as yet: the use of any particular coordinate system does not in itself restrict the type of mean flow considered, provided averaging is defined in a coordinate-independent way (A.M. § 2.1).

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When ϕ is velocity $\boldsymbol{u}$, the Stokes correction $\bar{\boldsymbol{u}}^S$ is sometimes referred to as the Stokes 'drift'. A typical illustration of the Stokes-drift effect is furnished by the parcel trajectory shown in figure 1. This was calculated for a hypothetical, oscillatory fluid motion whose Eulerian-mean velocity $\bar{\boldsymbol{u}}$ is exactly zero: the mean velocity following the parcel is, by contrast, quite different from zero.

To define ξ for small-amplitude waves, it is customary to resort to the simplest possible approximation

$$\frac{\partial}{\partial t}\,\xi(\boldsymbol{x},t) = \boldsymbol{u}'(\boldsymbol{x},t), \tag{6}$$

(see, for example, Longuet-Higgins 1969, equation (2)), where we have defined, again bearing subsequent generalization in mind,

$$\boldsymbol{u}'(\boldsymbol{x},t) \equiv \boldsymbol{u}(\boldsymbol{x},t) - \overline{\boldsymbol{u}(\boldsymbol{x},t)} \tag{7}$$

(the familiar Eulerian measure of disturbance velocity), and where

$$\overline{\xi(\boldsymbol{x},t)} = 0. \tag{8}$$

The last relation formally expresses the idea that ξ should be a *disturbance-associated* quantity.

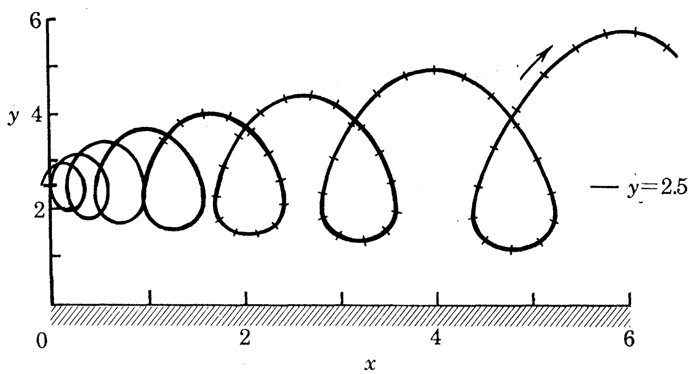

FIGURE 1. Path of a single fluid parcel in the two dimensional velocity field $u = 0.01\,t\cos(x-t)$, $v = 0.01\,yt\sin(x-t)$, satisfying the incompressibility condition $u_x+v_y = 0$. At $t = 0$ the parcel is at $x = 0$, $y = 2.5$; its subsequent trajectory was computed by means of a fourth order Runge–Kutta algorithm. Part of the path is marked at equal time intervals $\Delta t = 0.5$.

Since equation (6) is a differential equation, some kind of initial condition or equivalent information must be prescribed. When $\overline{(\ )}$ is a time average the condition (8) is sufficient to provide that information. But when $\overline{(\ )}$ is a space or ensemble average, (8) is *not* sufficient, because (6) and (8) are still satisfied if we add to ξ any time-independent function $\boldsymbol{f}(\boldsymbol{x})$ with zero mean ($\bar{\boldsymbol{f}} = 0$). This point is not usually discussed, because in classical wave theories there is usually no point in taking $\overline{(\ )}$ as anything other than a time mean, to within the approximations involved in those theories and in equation (6).

It can be shown that the approximations involved in using these ideas in their classical form require, in fact, that

(i) the disturbance amplitude a is small, in the sense that each component of ξ must be much smaller in magnitude than the corresponding smallest length scale of the disturbance; and

(ii) the mean velocities (Eulerian and Lagrangian) are both very small, formally $O(a^2)$ as $a \to 0$. (Thus the last term in (7) is actually irrelevant to the discussion so far.)

Although these requirements are sometimes satisfied in practice, it would seem desirable to have exact definitions applicable to finite-amplitude disturbances on arbitrary mean flows. Such definitions are provided by the generalized Lagrangian-mean theory, which can be seen as a natural generalization of the classical theory if we accept as natural the following three ideas. The first is that we should retain the relations (4), (5) and (8) exactly as they are, for analytical as well as conceptual simplicity. The second is the very reasonable idea that the natural Lagrangian measure of 'disturbance velocity' is not the right-hand side of equation (7) but, rather, a similar expression in which the last term is the *Lagrangian* mean, and the first term on the right is evaluated at the *displaced* position $\boldsymbol{x}+\boldsymbol{\xi}$. The resulting 'Lagrangian disturbance velocity' will be denoted by $\boldsymbol{u}^l$; by definition

$$\boldsymbol{u}^l(\boldsymbol{x}, t) \equiv \boldsymbol{u}\{\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x}, t), t\}-\overline{\boldsymbol{u}(\boldsymbol{x}, t)}^{\mathrm{L}}. \tag{9}$$

Note that this definition together with (4) (with $\boldsymbol{u}$ substituted for ϕ)) implies that $\overline{\boldsymbol{u}^l} = 0$, so that $\boldsymbol{u}^l$ is a disturbance-associated quantity, like $\boldsymbol{u}'$ or $\boldsymbol{\xi}$. Other Lagrangian disturbance quantities ϕ^l are defined similarly.

The third idea, which is obviously necessary if we want to remove the restriction (ii), and obtain a theory which behaves sensibly in a moving frame of reference, is that $\partial/\partial t$ in equation (6) should be replaced by the rate of change following the mean flow; and it is natural in this context to use the *Lagrangian*-mean flow. Putting the last two ideas together, we replace (6) by

$$\left(\frac{\partial}{\partial t}+\bar{\boldsymbol{u}}^{\mathrm{L}}\cdot\nabla\right)\boldsymbol{\xi} = \boldsymbol{u}^l, \tag{10}$$

where ∇ is the three dimensional gradient operator. Equations (4), (8), (9), and (10) are the basic equations of the generalized Lagrangian-mean theory. They assert that for each type of Eulerian-mean operator $\overline{(\ )}$, whether the averaging is done with respect to time, space, or an ensemble of realizations, there is a corresponding Lagrangian-mean operator $\overline{(\ )}^{\mathrm{L}}$. Once again, (10) is a differential equation and so suitable initial (or 'upstream') conditions, actual or hypothetical, are required to complete the definition, as discussed in A.M. (see their postulate (*viii*), p. 617). The simplest such conditions specify an initial state of no disturbance anywhere.

At first sight equations (4), (8), (9), and (10) look frighteningly implicit, but it is not hard to see that in the particular case where $\overline{(\ )}$ is a zonal average their content can be visualized with the aid of the following *mechanical analogy* (A.M. § 2.2). The analogy as we shall first describe it applies to the case where the Eulerian averaging operator $\overline{(\ )}$ is a rectilinear spatial average, parallel to the x axis, say, of some Cartesian coordinate system. We initialize the theory by assuming zero disturbance everywhere at $t = t_0$, say, and fix attention on a row R_0 of marked fluid parcels which are initially spaced at equal distances Δx along a line parallel to the x axis (figure 2*a*). We then watch those parcels as they follow the fluid motion. In the mechanical analogy (whose connection with the fluid motion is only kinematical, not dynamical) we imagine that a massless, rigid rod R initially coincides with R_0, but is subsequently free to move while constrained to remain parallel to the x axis. The position P of a typical parcel of fluid whose initial position was P_0 is joined to the point P_{R} on R, which initially coincided with P_0, by an idealized 'elastic band' $P_{\mathrm{R}}P$. This elastic band is also massless, and is such that P_{R} is pulled towards P (whose position is known from the given fluid motion) with a force just proportional to the distance $P_{\mathrm{R}}P$. Identical elastic bands similarly join all the other parcels to their corresponding points on the rod R, which is imagined to be in static equilibrium under the pull of all the elastic bands (and will therefore follow the parcels if they all happen to have a systematic motion in some direction).

Now if $\boldsymbol{x}$ is the current position of P_{R} at time t, $\boldsymbol{\xi}(\boldsymbol{x}, t)$ is the 'elastic band vector' $\overrightarrow{P_{\mathrm{R}}P}$, in the limit $\Delta x \to 0$, and $\bar{\boldsymbol{u}}^{\mathrm{L}}(\boldsymbol{x}, t)$ is the velocity of the rod R. For, in the first place, equation (8) is satisfied since in the limit $\Delta x \to 0$ it expresses the vanishing of the resultant force on R.

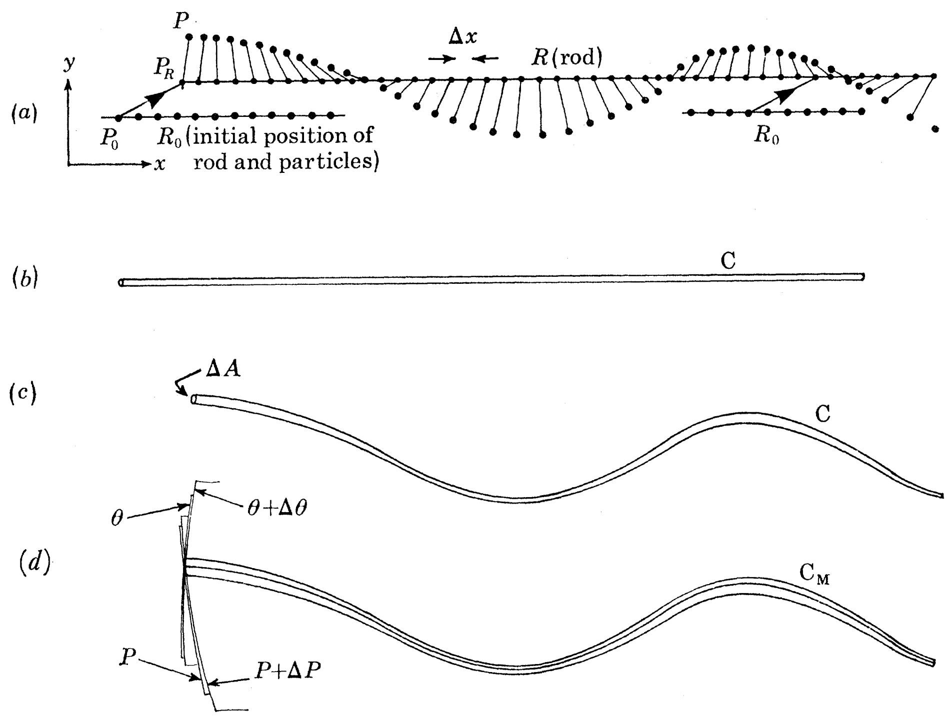

FIGURE 2. (*a*) Mechanical analogy for visualizing both the generalized Lagrangian mean and the finite-amplitude disturbance displacement function $\boldsymbol{\xi}(\boldsymbol{x}, t)$, in the case where averaging is spatial; see text below equation (10). (*b*) Uniform material tube C in a hypothetical initial state of no disturbance, corresponding to R_0 in (*a*). (*c*) Same material tube in the presence of a disturbance, giving domain of averaging for the spatial generalized Lagrangian mean (11). (*d*) Tube C_M marked out by surfaces of constant potential vorticity P and potential temperature θ.

(This depends on the assumed properties of our idealized elastic bands, and also on the equal spacing along R of the points $\boldsymbol{x}$ typified by P_{R} since the Eulerian averaging operator $\overline{(\;)}$ has uniform weighting in x.) In the second place, the properties of the elastic bands ensure that the rod's velocity will be just the average velocity of all the fluid parcels. But (4) with $\phi = \boldsymbol{u}$ states that the average velocity in this sense is $\bar{\boldsymbol{u}}^{\mathrm{L}}(\boldsymbol{x}, t)$, in the limit $\Delta x \to 0$, again because of the equal spacing of points along R. Finally, the remaining equations (9) and (10) are satisfied because (10) expresses the fact that $\boldsymbol{u}^l$ is equal to the rate of change of an individual elastic-band vector $\boldsymbol{\xi}$ such as $\overline{P_{\mathrm{R}}P}$, while (9) expresses the fact that this rate of change $\boldsymbol{u}^l$ is just the velocity of one end P of the elastic band relative to the other, P_{R}.

It was emphasized that the average given by equation (4) is an average along the line of parcels weighted by the number of parcels per unit length, when the parcels were *equally spaced along* R_0 in the initial state of no disturbance. This is equivalent to an average weighted by mass along a material tube C of fluid which was initially straight and uniform as suggested in figure 2*b*, and which has been distorted by the disturbance into the shape shown in figure 2*c*, with variable cross-sectional area ΔA. That is, the definition (4) is equivalent, in the present case of a Cartesian spatial average, to the alternative definition

$$\overline{(\ )}^{\mathrm{L}} \equiv \int_{\mathrm{C}} (\ )\, \rho \Delta A\, \mathrm{d}s \Big/ \int_{\mathrm{C}} \rho \Delta A\, \mathrm{d}s, \tag{11}$$

where $\mathrm{d}s$ is an element of arc length along C, ρ is density, and the cross-sectional area ΔA of the tube C is taken to be vanishingly small. It is important to realize that the non-uniform weighting of the average along C is essential if $\overline{(\ )}^{\mathrm{L}}$ is to correspond to the classical (unweighted) *time* mean following a single parcel, in situations where the classical approximations apply and where the Eulerian time and space means are equal. One such situation is discussed by Matsuno (1979).

An interesting corollary of the foregoing is that $\bar{\boldsymbol{u}}^{\mathrm{L}}$, besides being the velocity of the rod R in figure 2*a*, is also the velocity of the centre of mass of the tube C in figure 2*c* (A.M. § 4.3). The small-amplitude approximation to this exact result was previously noted by T. Matsuno (personal communication). From the centre-of-mass property we can deduce in turn that the Lagrangian-mean vertical velocity is just the rate of change of potential energy of the material tube C, per unit mass. It follows that the vertical Stokes drift, if defined exactly as in (5), can be regarded as an exact measure of the rate of change of eddy available potential energy; an explicit example will be noted in § 4.

2.2. *Small-amplitude approximations*

For disturbances of small amplitude a, Taylor expansion of (4) with respect to $\boldsymbol{\xi}$ and substitution into (5), making use of (8), shows that

$$\overline{\phi}^{\mathrm{S}} = \overline{\boldsymbol{\xi}\cdot\nabla\phi'} + \overline{\tfrac{1}{2}\boldsymbol{\xi}\cdot(\boldsymbol{\xi}\cdot\nabla)}\nabla\overline{\phi} + O(a^3), \tag{12}$$

where ∇ is the three dimensional gradient operator; for more details see A.M. § 2.4. Written out in Cartesian component form for the case of a rectilinear spatial average with respect to x, the second term becomes

$$\tfrac{1}{2}(\overline{\eta^2}\overline{\phi}_{yy} + \overline{2\eta\zeta}\overline{\phi}_{yz} + \overline{\zeta^2}\overline{\phi}_{zz}), \tag{13}$$

where $\boldsymbol{x} = \{x, y, z\}$ and $\boldsymbol{\xi} = \{\xi, \eta, \zeta\}$, and where subscripts denote partial differentiation. If in addition the flow is incompressible, the first term on the right of (12) may be rewritten

$$\overline{(\eta\phi')_y} + \overline{(\zeta\phi')_z} + O(a^3), \tag{14}$$

since for incompressible flow it may be shown that

$$\nabla\cdot\boldsymbol{\xi} = O(a^2) \tag{15}$$

(A.M. § 9). In all these small-amplitude formulae, including the general formula (12), we may approximate (10) by

$$(\partial/\partial t + \bar{\boldsymbol{u}}\cdot\nabla)\ \boldsymbol{\xi} = \boldsymbol{u}' + \boldsymbol{\xi}\cdot\nabla\bar{\boldsymbol{u}} + O(a^2). \tag{16}$$

The relation (12) reminds us of the well known fact that the Stokes correction $\overline{\phi}^{\mathrm{S}}$ is $O(a^2)$, that is to say second order in disturbance amplitude a, as $a \to 0$. The first term on the right of (12) represents the result originally derived by Stokes (1847); it expresses the correction to the Eulerian mean arising from displacements through disturbance-associated gradients.

Longuet-Higgins (1969) presents some alternative expressions for this term which hold in cases where all averaged quantities are time-independent. The second term of (12), a correction arising from displacements through non-uniform *mean* gradients of ϕ, is neglected in classical treatments (and is sometimes overlooked in cases where it is not negligible). It is easy to discern examples of meteorological interest where the effects described by this term will not be negligible, and one such example will be mentioned in § 5 below. Another such example is the general problem of the mean zonal acceleration due to equatorial planetary waves (A.M. § 9).

2.3. *Other averages arising in the generalized Lagrangian-mean theory*

The operator $\overline{(\ )}^{\mathrm{L}}$ is by no means the only type of average that arises in the g.l.m. theory; different averages are appropriate to different circumstances. For example the most natural measure of 'mean density' for the purpose of expressing mass conservation analytically turns out to be the quantity

$$\tilde{\rho}(\boldsymbol{x}, t) = \rho(\boldsymbol{x}+\boldsymbol{\xi}, t)\,\frac{\partial(x+\xi,\, y+\eta,\, z+\zeta)}{\partial(x, y, z)}, \tag{17}$$

where the last factor is a Jacobian determinant involving the derivatives of $\boldsymbol{\xi}$ with respect to the reference position $\boldsymbol{x}$:

$$\frac{\partial(x+\xi,\, y+\eta,\, z+\zeta)}{\partial(x, y, z)} = \begin{vmatrix} 1+\xi_x & \xi_y & \xi_z \\ \eta_x & 1+\eta_y & \eta_z \\ \zeta_x & \zeta_y & 1+\zeta_z \end{vmatrix}. \tag{18}$$

Despite the absence of an overbar on the right of (17), $\tilde{\rho}$ is indeed a mean quantity ($\bar{\tilde{\rho}} = \tilde{\rho}$) when the theory is properly initialized, as is shown in A.M. § 4. It is also shown there that $\tilde{\rho}$ satisfies

$$\partial\tilde{\rho}/\partial t + \nabla\cdot(\tilde{\rho}\bar{\boldsymbol{u}}^{\mathrm{L}}) = 0, \tag{19}$$

which, analytically speaking, is the simplest expression of mean mass conservation in the g.l.m. theory.

Another physically important mean quantity is

$$\begin{aligned} \boldsymbol{U}(\boldsymbol{x}, t) &\equiv \overline{\{\nabla(\boldsymbol{x}+\boldsymbol{\xi})\}\cdot\{\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi}, t) + \boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{\xi})\}} \\ &= \bar{\boldsymbol{u}}^{\mathrm{L}} + \boldsymbol{\Omega}\times\boldsymbol{x} + \overline{(\nabla\boldsymbol{\xi})\cdot(\boldsymbol{u}^{l} + \boldsymbol{\Omega}\times\boldsymbol{\xi})}, \end{aligned} \tag{20}$$

where the dot denotes scalar multiplication operating on $\boldsymbol{x}$ and $\boldsymbol{\xi}$, not ∇, and where a frame of reference rotating with constant angular velocity $\boldsymbol{\Omega}$ is assumed. $\boldsymbol{U}$ is the basic measure of 'absolute' mean velocity underlying the 'non-acceleration theorems' and their generalizations of A.M. § 3. When $\overline{(\ )}$ is a rectilinear spatial average with respect to x, the x component U_1 of $\boldsymbol{U}$ is readily expressed in an alternative form paralleling (11),

$$U_1 = \int_{\mathrm{C}} (\boldsymbol{u} + \boldsymbol{\Omega}\times\boldsymbol{x})\cdot \mathrm{d}\boldsymbol{s} \Big/ \int_{\mathrm{R}} \mathrm{d}x; \tag{21}$$

i.e. U_1 is Kelvin's *circulation* along C, divided by the length $\int \mathrm{d}x$ of the corresponding rod R in figure 2*a* (with the limit of infinitely long R and C understood if $\overline{(\ )}$ is a non-periodic, rectilinear spatial average). The key role of Kelvin's circulation function in the theory of wave, mean-flow interaction was recognized by Rayleigh (1896, § 352), in the different context of acoustics, and its importance for all kinds of oceanic and atmospheric waves (stemming from *Bjerknes's* circulation theorem) appears to have been first recognized by Bretherton

(1969 *b*, 1971). A study of the behaviour of the circulation function itself (the numerator of (21)) in the wintertime stratosphere might be of great interest. Use of the circulation function avoids certain problems in computing Lagrangian-mean angular momentum budgets, the eddy contributions to which involve the pressure or height field (Bretherton 1969*a*, 1971, 1979; A.M. § 8; Uryu 1979; Dunkerton 1979) in such a way as to make accurate computation difficult. The budget of the circulation function, by contrast, does not involve the pressure field directly when C lies in an isentropic surface.

A further measure of mean motion of physical interest, especially for incompressible flow or for hydrostatic flow described in pressure coordinates, is the vector field $\boldsymbol{V}(\boldsymbol{x}, t)$ whose jth cartesian component is given by

$$V_j = \sum_{i=1}^{3} \overline{u_i(\boldsymbol{x}+\boldsymbol{\xi}, t)\, K_{ij}}. \tag{22}$$

Here K_{ij} is the two-by-two determinant consisting of the Jacobian (18) with its ith row and jth column removed. $\boldsymbol{V}$ has a natural interpretation as a measure of mean volume flux. When $\overline{(\ )}$ is a rectilinear spatial average with respect to x, the second or y component V_2 of $\boldsymbol{V}$, when multiplied by a small length Δz, represents the rate at which fluid volume crosses a fixed, ribbonlike surface $R_{\Delta z}$ whose edges momentarily coincide with the two curves $C(y, z, t)$ and $C(y, z+\Delta z, t)$ which correspond to rods R momentarily at (y, z) and $(y, z+\Delta z)$. More precisely,

$$V_2\,\Delta z = \int_{R_{\Delta z}} \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S \Big/ \int_{R} \mathrm{d}x, \tag{23}$$

where $\boldsymbol{n}$ is a unit normal to $R_{\Delta z}$ oriented toward the side corresponding to positive y, $\int_R \mathrm{d}x$ is the length of rod corresponding to $R_{\Delta z}$ and the limit $\int_R \mathrm{d}x \to \infty$ is understood where necessary, as before. The relation (23) is an immediate consequence of equation (A 15) of A.M. A similar relation gives $V_3\,\Delta y$ with $R_{\Delta y}$ in place of $R_{\Delta z}$. The relation between $\boldsymbol{V}$ and $\bar{\boldsymbol{u}}^{\mathrm{L}}$ is not very direct, in that the latter does not give the rate at which fluid crosses the surfaces $R_{\Delta z}$, $R_{\Delta y}$ when those surfaces are held fixed in space: rather, the y and z components of $\tilde{\rho}\bar{\boldsymbol{u}}^{\mathrm{L}}$ give the rate at which fluid mass crosses the same surfaces when z, $z+\Delta z$ and y, $y+\Delta y$ are held fixed in the *reference* space of rod positions, in which case the surfaces $R_{\Delta z}$ and $R_{\Delta y}$ will generally be moving. Related to these properties are the facts that for an incompressible flow $(\nabla\cdot\boldsymbol{u} = 0)$ we generally have

$$\nabla\cdot\boldsymbol{V} = 0 \quad \text{but} \quad \nabla\cdot(\tilde{\rho}\bar{\boldsymbol{u}}^{\mathrm{L}}) \neq 0. \tag{24}$$

The first relation is easy to verify directly from (22), using equations (A 3) and (A 4) of A.M. [respectively $\Sigma\,\partial K_{ij}/\partial x_j = 0$, and a result equivalent to the chain rule for differentiating a function of $\boldsymbol{x}+\boldsymbol{\xi}(\boldsymbol{x}, t)$]. The second relation of (24), a consequence of (17) and (19), is a property of the classical Lagrangian mean as well as its generalizations (see, for example, McIntyre 1973). It is discussed further in § 4 below.

3. Re-initialisation

We have emphasized the fact that the concept of 'Lagrangian mean' and the definitions of related mean quantities generally depend on a suitable initialization of the theory, implicit or explicit. One cannot normally speak of a unique generalized Lagrangian mean, but only of a generalized Lagrangian mean relative to the initialization used. One important exception, when the averaging is spatial, is strictly conservative motion in the case where the material tubes C can be defined by the intersection of surfaces of constant potential vorticity P and potential temperature θ, as suggested in figure 2*d* and discussed further in A.M. (appendix C). For the real stratosphere, however, P and θ are not precisely conserved, and we must expect the shapes of tubes C to become less and less closely related to the wave motion as time goes on. Such temporally non-uniform behaviour is a basic feature of any description of real, dissipative fluid motion using Lagrangian ideas, and it is clearly an essential part of how the 'non-acceleration' and 'non-transport' constraints are broken by real fluid motion. (In particular, these constraints can be broken catastrophically by large-amplitude wave events, such as those associated with sudden warmings (J. D. Mahlman, C.-P. Hsu, personal communication), which irreversibly coil parts of the tubes C into complicated shapes. We may aptly speak of 'breaking' planetary waves in such cases.)

As a practical matter, therefore, some kind of continuous or repeated *re-initialization* of the theory will be necessary if averages like (11), (21) or (23) are to be useful in numerical or observational diagnostics. Moreover, the most appropriate procedure would depend on the problem at hand. One possibility, when disturbance amplitudes are not too large, is simply to replace (11) by a 'modified Lagrangian mean' operator

$$\overline{(\ )}^{\mathrm{M}} \equiv \int_{\mathrm{C_M}} (\ )\, \rho \Delta A_{\mathrm{M}} \,\mathrm{d}s \Big/ \int_{\mathrm{C_M}} \rho \Delta A_{\mathrm{M}} \,\mathrm{d}s, \tag{25}$$

where ΔA_{M} is the cross-sectional area of the tube $\mathrm{C_M}(P, \theta)$ of figure 2*d*. This tube is bounded by the pair of constant-P surfaces whose potential vorticities are P and $P+\Delta P$, and the pair of isentropes whose potential temperatures are θ and $\theta+\Delta\theta$, where ΔP and $\Delta\theta$ are small. The idea of using $\mathrm{C_M}$ in this way arose in conversations with T. Dunkerton. $\overline{(\ )}^{\mathrm{M}}$ is the same as $\overline{(\ )}^{\mathrm{L}}$ when P and θ are conserved; but we are at liberty to define and use $\overline{(\ )}^{\mathrm{M}}$ whether or not equation (1) is satisfied, knowing that the whole theoretical structure resulting from its use will be the same as that of the basic generalized Lagrangian-mean theory apart from terms representing departures from frictionless, adiabatic motion. Moreover, the shape of $\mathrm{C_M}$, being determined by the fields of potential vorticity and potential temperature, has no systematic tendency to become unrelated to the wave motion as time goes on; therefore $\overline{(\ )}^{\mathrm{M}}$ will not exhibit the non-uniform behaviour in time which frictional and diabatic effects must generally induce in ξ and therefore $\overline{(\ )}^{\mathrm{L}}$. Such a continuously re-initialized theory would apply as long as surfaces of constant P and θ continued to intersect at finite angles. Note that dissipative terms must now appear on the right of equation (10) in particular; the correspondingly modified ξ function will be denoted by ξ^{m}. It is interesting that the operator $\overline{(\ )}^{\mathrm{M}}$ appears implicitly in the quite different but also fundamental context of constructing 'integrals of the motion', in the sense of classical dynamics (Obukhov 1971).

If disturbance amplitudes become too large, then the tube $\mathrm{C_M}$ may change its topology by splitting into more than one piece; this certainly happens for the breaking planetary waves involved in major warmings. The corresponding ξ^{m} will then behave discontinuously. Examples in the troposphere are also well documented (see, for example, Obukhov 1964; Danielsen 1968). To study isolated events of such large amplitude it may prove more useful to pick an initial time t_0 at which the tubes $\mathrm{C_M}$ have simple shapes, if that is possible, and then follow the material tubes C which coincided with the tubes $\mathrm{C_M}$ at time t_0 (Dunkerton 1979). This idea, together with the arguments of A.M. appendix C, could be made the basis of an initialization procedure satisfying the conditions assumed by A.M. (particularly their postulate (viii), p. 617), and avoiding the discontinuous behaviour of ξ^{m}. Even though the initial state of no disturbance envisaged by A.M. may never have existed, the procedure would define a ξ which would coincide with ξ^{m} at time t_0 – i.e. which would correspond to the shapes of the tubes C_M at time t_0 – and which therefore could have evolved, via conservative motion, from a *hypothetical* initial state of no disturbance.

4. Some further properties of generalized Lagrangian means

4.1. *The divergence effect*

In § 2.3 we noted that $\bar{\boldsymbol{u}}^{\mathrm{L}}$ is generally divergent even for incompressible flow. Since this fact is sometimes regarded with surprise, it may be worth pointing out (following A.M. § 9) how plausible it becomes as soon as we consider flow near a boundary. Take the case of spatial averaging again, and suppose that the tube C in figure 2*b* is initially parallel to and very close to a wall y = constant. Then if a disturbance grows and C begins to look more like figure 2*c* its centre of mass will usually move away from the wall; a similar effect is of course familiar in problems of turbulent diffusion near boundaries.

It should be recognized that the same phenomenon occurs for the classical Lagrangian time mean following a single fluid parcel, as well as for its various generalizations. This is clear, for instance, from the illustration given in figure 1, depicting the trajectory of a single parcel computed for the incompressible velocity field defined in the figure caption. That velocity field has zero normal component at the boundary shown, and represents an oscillatory disturbance whose amplitude is uniform with distance parallel to the wall, but growing in time. The parcel's position at equal time intervals is marked off along part of the trajectory. It is clear by inspection of these marks that, as time goes on, the parcel spends more time further away from the wall, so that a running time average of the parcel's position over several cycles will show a systematic motion away from the wall. Since in the given velocity field all mean quantities are independent of distance parallel to the wall, *all* fluid parcels initially lying on the plane $y = 2.5$ (from which the computed trajectory begins) will have a mean motion away from the wall. The classical Lagrangian-mean motion is evidently divergent; and it can be shown directly from (12) that the effect is $O(a^2)$ and therefore comparable with all other mean effects of the waves even when their amplitude a is arbitrarily small (e.g. McIntyre 1973).

The divergence effect is further illustrated by a recent calculation (Uryu 1979) of the generalized Lagrangian-mean meridional circulation for the Eady model of baroclinic instability in a channel. In this case the average used is the spatial g.l.m. given by equation (11). Uryu's result is reproduced as figure 3; again, the model flow is strictly incompressible. Nevertheless, $\bar{\boldsymbol{u}}^{\mathrm{L}}$ diverges away from the side walls as the disturbance grows; there is a corresponding tendency for convergence in mid-channel. Another feature of interest to note in passing is the fact that the vertical motion is thermally direct, in complete contrast to the Eulerian-mean motion which as is well known has the classical three-cell structure, with a large indirect cell in the middle (Uryu 1979). The thermal directness is reminiscent of results obtained for actual laboratory and atmospheric flows by Riehl & Fultz (1957), Krishnamurti (1961), Danielsen (1968, figure 14), and Mahlman (1969) even though the quasi-Lagrangian averaging procedures of those investigators do not correspond exactly to any of those defined here, if only because the averages were weighted differently. The thermal directness of the Lagrangian-mean vertical motion in figure 3 reminds us that the growing Eady wave is reducing the total potential energy of the system. Note that the potential energy change due to the Lagrangian-mean motion (the motion of the centres of mass of all the tubes C) is the total potential energy change – there is no additional 'eddy' contribution. What is usually called the rate of change of eddy available potential energy appears implicitly as the contribution associated with the Stokes drift, as was mentioned earlier.

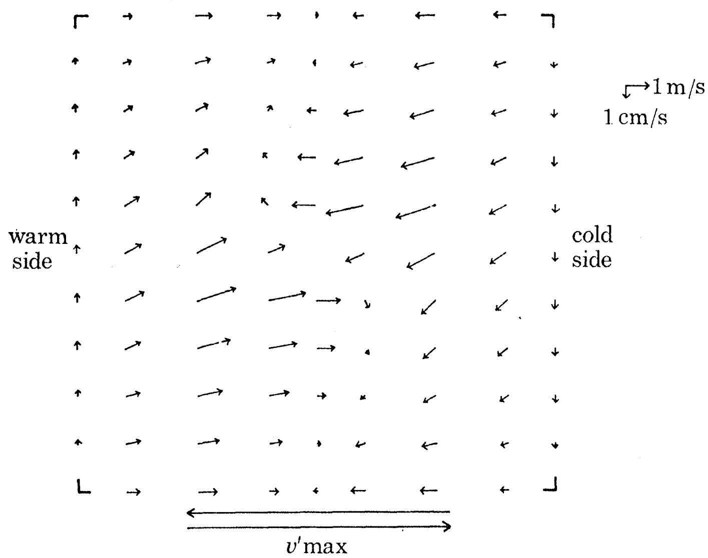

FIGURE 3. Generalized Lagrangian-mean meridional circulation $\bar{v}^L$, $\bar{w}^L$ associated with a growing Eady wave, from Uryu (1979). The fluid motion is incompressible, but the Lagrangian-mean motion is divergent because of disturbance growth (see text). Parameter values are as follows. Disturbance amplitude, as measured by maximum north–south Eulerian disturbance velocity v', 11 m s^{-1} (shown on the same scale as $\bar{v}^L$ by the long arrows at bottom); growth rate 0.7 day^{-1}; wavelength 5000 km; width of channel (meridional *half*-wavelength) 5000 km; height of channel 10 km; vertical shear of basic flow 3 m s^{-1} km^{-1}; buoyancy (Brunt–Väisälä) frequency 10^{-2} s^{-1}; Coriolis parameter 10^{-4} s^{-1}.

As we saw from equation (24) one can define, for incompressible flow (or for hydrostatic flow viewed in pressure coordinates), a mean circulation which is *not* divergent but which is nevertheless closer to Lagrangian concepts than is the standard Eulerian mean. Such a mean motion would not correspond to the classical Lagrangian mean, even in the crudest approximation; and it does not have such a simple relationship with potential energy. Nor does it have such simple analytical properties as $\bar{\boldsymbol{u}}^L$; nevertheless, it might prove useful, and further investigation is warranted.

In all our examples (and the same is true of the example given in McIntyre 1973) the divergent character of $\bar{\boldsymbol{u}}^L$ stems from temporal changes in the amplitude of the disturbance. In the cases considered, $\nabla\cdot\bar{\boldsymbol{u}}^L$ is attributable in the first approximation to the term $\partial\tilde{\rho}/\partial t$ in equation (19). It should not be thought, however, that this term is necessarily zero in all problems for which the wave amplitude is steady. When the fluid behaviour is dissipative, the temporal non-uniformity mentioned earlier can cause any quantity depending on ξ, including $\tilde{\rho}$ in equation (19), to change systematically in time. The effect is formally negligible in certain problems where the wave dissipation is assumed small, but not in general. These remarks are of interest in showing how the present ideas fit in with the results of Kida (1977) on the behaviour of initially zonal lines of air parcels in a general circulation model of the atmosphere. As could have been anticipated from our discussion (see also equation (29) below), Kida found a strong divergence effect in his model troposphere (with a pattern rather like that in figure 3) even though the total eddy field was not far from statistically steady.

One consequence of the divergence effect, for quasi-geostrophic motions such as the Eady wave, is that it can upset the standard quasi-geostrophic scaling for the mean flow. For instance the Lagrangian-mean meridional velocity $\bar{v}^L$ in the example of figure 3 exceeds its Eulerian-mean counterpart $\bar{v}$ by a large factor of the order of the inverse Rossby number. This warns us that standard order-of-magnitude arguments may need re-examination when a Lagrangian-mean description is used; further illustrations of this point will be found in A.M. § 9 and in the cited paper by Uryu. An associated phenomenon is that eddy terms involving the disturbance pressure field in the Lagrangian-mean momentum or angular momentum budget (see references cited below equation (21)) tend to be correspondingly large and to oppose the Coriolis force associated with $\bar{v}^L$ (Uryu 1979). This is one reason why it was suggested earlier that a direct study of the Kelvin–Bjerknes circulation function (the numerator of equation (21)) might prove more useful in practical diagnostics than the angular momentum itself, important though the latter may be conceptually.

4.2. *Some difficulties in using* $\overline{(\;)}^L$ *on the sphere*

For use in connection with realistic models of the stratosphere, the generalized Lagrangian-mean theory will need to be applied on the sphere. Unfortunately this presents some additional difficulties of a rather peculiar kind, for waves of finite amplitude. When $\overline{(\;)}$ is a zonal average on the sphere, the basic prescription of § 3 can be expressed by means of a mechanical analogy similar to that of figure 2*a*; but in place of the rigid rod R constrained to remain parallel to the *x*-axis, we have to imagine a frictionless, massless 'magic hoop' constrained by marvellous machinery to remain symmetrically disposed about the Earth's axis, while remaining free to expand and contract radially, to move axially, and to rotate about the axis. The positions of air parcels are imagined to be joined to this magic hoop by ideal elastic bands as before. The motion of the hoop under the balanced pull of all the elastic bands then gives the vector $\bar{\boldsymbol{u}}^L(\boldsymbol{x}, t)$ at each position $\boldsymbol{x}$ on the hoop. This analogy expresses the content of equations (4), (8), (9) and (10) when $\overline{(\;)}$ is the Eulerian zonal mean on the sphere.† The equivalence between equations (4) and (11) now follows just as before, although the interpretation of equation (11) when a vector quantity is to be averaged requires caution as will be seen shortly. Equation (21) involving the circulation function also holds, provided we interpret U_1 as the zonal component of $\boldsymbol{U}$, and the denominator on the right as the circumference of the magic hoop H.

Now the smallness of the atmosphere's scale height in comparison with the Earth's radius forces planetary-scale parcel excursions to take place along curved, quasi-horizontal paths closely following the shape of the Earth's surface. Consider for example the extreme case sketched in figure 4, in which a hypothetical tube C and its associated magic hoop H are shown. (The elastic bands joining the two are invisible.) Clearly the magic hoop will be pulled *underground* whenever the disturbance displacement attains any substantial amplitude. Now a reference position $\boldsymbol{x}$ located underground does not in itself detract from the power of

† To verify the correctness of the analogy mathematically, it is best to begin with a coordinate-independent statement of what is meant by the zonal average of a vector. A suitable statement is given at the bottom of p. 614 of A.M.; see also the end of p. 622.

the generalized Lagrangian-mean theory to reveal the theoretical generalities of wave, mean-flow interaction – but it is distinctly inconvenient for making mental or actual pictures of specific mean flows. It is, however, an inevitable consequence of requiring that $\boldsymbol{\xi}$ satisfy (8), in a true vector sense independent of particular coordinate systems; and the *analytical* power of the g.l.m. theory seems to depend upon (8).

A related and more serious difficulty is that the tubes C and their sets of reference positions $\boldsymbol{x}$ (lying on the hoops H) are not necessarily in one-to-one correspondence: two C's at different heights in the atmosphere could evidently have the same H. This could happen whenever the disturbance amplitude increased with height at a certain rate, depending on disturbance amplitude. In such cases, which are presumably common enough, the Jacobian (18) and therefore the mean quantity $\tilde{\rho}$ would become infinite somewhere. (So would the mean quantities $\boldsymbol{U}$ and $\boldsymbol{V}$ defined in § 2.3, with the important exception of the zonal component of $\boldsymbol{U}$, in virtue of equation (21); only $\bar{\boldsymbol{u}}^{\mathrm{L}}$ would remain finite in all three components.)

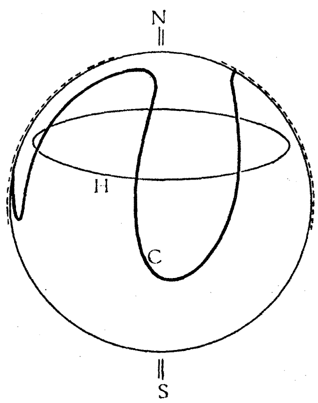

FIGURE 4. Tube C in the presence of a hypothetical, large-amplitude disturbance (not intended to look realistic). The fine dashed lines indicate projections of C onto meridional cross sections. The 'magic hoop' H gives the set of reference positions $\boldsymbol{x}$ associated with C, in terms of which the generalized Lagrangian-mean theory takes the fundamental, coordinate-independent form implied by equations (4), (8), (9), and (10).

One possible way of avoiding this singular behaviour of the reference space would be to use mean latitude ($\bar{y}^{\mathrm{L}}$ or $\bar{\phi}^{\mathrm{L}}$) and height ($\bar{z}^{\mathrm{L}}$) or pressure ($\bar{p}^{\mathrm{L}}$), defined by equation (11), as reference coordinates in place of $\boldsymbol{x}$. The true vector reference position $\boldsymbol{x}$ would still be needed to define the vector $\boldsymbol{\xi}$ (by locating the back end of the corresponding elastic band), but $\boldsymbol{x}$ would no longer be used as the independent variable as well. Part of the price would be loss of the analytical simplicity, as well as the coordinate-independent form, of equation (19). Another alternative which might sometimes be feasible would be simply to use P and θ as coordinates, as suggested by equation (25).

A corollary of the magic-hoop kinematics suggested by figure 4 is that the vertical component of the true vector Lagrangian-mean velocity $\bar{\boldsymbol{u}}^{\mathrm{L}}$ will be large and negative for a rapidly growing disturbance, purely because of the combined effects of disturbance transience and the Earth's curvature. Lest the reader think that such bizarre behaviour is peculiar to the spatial Lagrangian mean under discussion, it should be pointed out that the same thing happens for the classical time mean following a single parcel. If as in figure 1 we consider the behaviour of a single parcel in a growing disturbance, we can see at once that if the disturbance is of the type suggested in figure 4, involving large north–south excursions, then a running time average of the velocity vector for the parcel, taken over several oscillations, will have a large component toward the centre of the earth just as before. The result of applying the operator $\overline{(\ )}^{L}$ to the vertical *component* of velocity will, by contrast, have the smaller order of magnitude which is characteristic of the actual large-scale vertical motion, and may therefore be a more useful quantity to deal with in meteorological practice. It is this quantity, incidentally, rather than the vertical component of the vector $\bar{\boldsymbol{u}}^{L}$, that is relevant to potential energy in a spherical gravitational potential Φ, the potential energy of the tube C being the scalar quantity $\bar{\Phi}^{L}$, per unit mass.

It is evident from what has just been said that, whenever $\overline{(\ )}$ is a curvilinear spatial average, the operation of taking the curvilinear components of a vector does not generally commute with the operation of taking its generalized Lagrangian mean $\overline{(\ )}^{L}$.† It can, however, be shown that when $\overline{(\ )}^{L}$ is applied separately to the latitudinal and vertical components of any vector $\boldsymbol{v}$ in spherical coordinates the result is the 'contravariant components' of $\bar{\boldsymbol{v}}^{L}$ in the sense of general tensor analysis, with respect to a coordinate system in which $\bar{y}^{L}$ and $\bar{z}^{L}$ are regarded as coordinates defining the position of the magic hoop. (The correspondence between $\boldsymbol{x}$ and $(\bar{y}^{L}, \bar{z}^{L})$ is now being looked upon as a coordinate transformation.) It is beyond the scope of this paper to go further into the technicalities of general tensor analysis; the main point to be borne in mind is that *the meaning of such concepts as 'Stokes drift' depends on whether the concept is applied to the velocity vector or only to one of its spherical components.* It is the latter version of the concept that will probably be used most in practice, for reasons already mentioned, and also because it fits in best with the familiar parallelism between spherical formulations and channel or beta-plane formulations. This parallelism allows us to think of spherical coordinates as approximately Cartesian and thus to sidestep the issues raised by figure 4, as is done for example in A.M. § 9, in Matsuno & Nakamura (1979), in Matsuno (1979) and in McIntyre (1979).

Another place where the Jacobian (18) may become singular is near the poles; the region of misbehaviour can be expected to increase in size as disturbance amplitude increases. It is obvious, indeed, that there can be no such thing as a zonally asymmetric set C of air parcels with a mean latitude of 90°. It should be remembered, however, that in the presence of large amplitude disturbances to the zonal flow the operation of taking any kind of zonal average near the pole, while well-behaved mathematically in the Eulerian case, as it happens, is certainly quite meaningless physically. Thus the refusal of the generalized Lagrangian-mean theory to behave itself near the poles, or for that matter to assign any mean values at all to some locations near the poles, could well be regarded as a desirable property of the theory rather than the reverse. A theory which automatically recognizes a domain where zonal averaging becomes physically meaningless should perhaps be regarded as a more 'intelligent' theory than one which does not.

The use of potential vorticity P and potential temperature θ as coordinates may avoid some of the foregoing problems. Equation (25) becomes relevant, in any case, whenever we want to use $\overline{(\ )}^{M}$ instead of $\overline{(\ )}^{L}$ so as to obtain a continuously re-initialized theory. However, breakdown of one-to-one correspondence between pairs of values of (P, θ) and quasi-material tubes C_M might still have to be coped with, since a surface of constant P might intersect a surface of constant θ more than once. Also, a practical disadvantage of P and θ as coordinates is the

† We are taking the fundamental definition of $\overline{(\ )}^{L}$ as equation (4), of course, since equation (11) has no useful coordinate-independent meaning when operating on a vector.

fact that the surfaces of constant P in the extratropical stratosphere tend to be nearly horizontal, and are therefore not usually at a large angle to those of constant θ. This occurs at any rate when P is given its usual definition in pressure coordinates, namely isentropic absolute vorticity times $\partial\theta/\partial p$. The main reason for the strong vertical gradient in P is the roughly exponential fall-off of pressure p with height and the corresponding gradient in $|\partial\theta/\partial p|$. However, whenever P is conserved, so also is the related quantity P_{F} formed by replacing θ by an arbitrary function $F(\theta)$ in the definition of P; judicious choice of $F(\theta)$ could certainly give a modified potential vorticity P_{F} with a far weaker vertical gradient and therefore more convenient for use as a coordinate with θ.

Other ways around the various difficulties will no doubt be thought of in due course. Again, it will not be clear how useful the foregoing suggestions will be until more experience is gained, probably in the first place with the results of numerical models. It is hoped, as was said before, that the present, very preliminary, discussion might help stimulate the necessary thinking and experimentation.

5. Concluding remarks

An important topic, as yet little investigated, is how best to express the effects of diabatic heating and other departures from conservative motion on the various mean quantities of interest. These effects are the interesting ones for applications to the real atmosphere, and estimates of diabatic and frictional tendencies in the potential vorticity P from the work of Staley (1960), Danielsen (1968) and others will clearly be essential background, in addition to estimates of radiative and turbulent diabatic heating *per se*.

If equation (1) is replaced by

$$\mathrm{D}X/\mathrm{D}t = Q_X, \tag{26}$$

then (3) becomes

$$\left(\frac{\partial}{\partial t}+\bar{v}^{\mathrm{L}}\frac{\partial}{\partial y}+\bar{\omega}^{\mathrm{L}}\frac{\partial}{\partial p}\right)\bar{X}^{\mathrm{L}} = \bar{Q}^{\mathrm{L}}_X. \tag{27}$$

(A.M., equation 2.21). When $X = \theta$ then $Q_X, = Q$ say, is the total diabatic heating rate expressed as potential temperature change per unit time, and $\bar{Q}^{\mathrm{L}}$ is its generalized Lagrangian mean (as defined for instance by equation (11) above). Dunkerton (1978), using small-amplitude theory for steady waves (and thus avoiding the worst difficulties mentioned in § 4) has argued that for qualitative purposes we may replace $\bar{Q}^{\mathrm{L}}$ on the right-hand side of (27) by $\bar{Q}$, the Eulerian-mean heating (but *not* $\bar{\omega}^{\mathrm{L}}$ by $\bar{\omega}$ on the left). That is, the Stokes correction $\bar{Q}^{\mathrm{S}}$ can be neglected in comparison to $\bar{Q}$ and $\bar{Q}^{\mathrm{L}}$ (although the Stokes correction $\bar{\omega}^{\mathrm{S}}$ cannot). However, the argument must ultimately rely on numerical estimates of net radiative heat-flux divergences $Q_{(r)}$ and turbulent heat-flux divergences $Q_{(t)}$, matters of some uncertainty. It perhaps ought to be remarked that near the bottom of the stratosphere, where turbulent mixing is important at tropopause-folding events (for recent work on this see Shapiro 1978), the correlation between those events and large-scale weather patterns, and therefore with the shapes of material tubes C, could easily make $\overline{(Q_{(t)})}^{\mathrm{S}}$ a significant contribution to $\bar{Q}^{\mathrm{L}}$. (Dunkerton's discussion was, however, concerned less with the bottom of the stratosphere than with the middle atmosphere as a whole.)

At the qualitative level of approximation used by Dunkerton, the temporally non-uniform behaviour of the ξ function (§ 3 above) does not play a significant role in the discussion. For a more quantitative analysis, however, it will be essential to introduce re-initialization procedures (§ 3), and to take account of the fact that dissipative processes make the divergence effect (§ 4.1) significant even for steady waves. For example, in the case of spatial averaging, and in a situation where the modified Lagrangian mean defined by (25) can be used, it is evident from equation (27) that

$$\left(\frac{\partial}{\partial t}+\bar{v}^{\mathrm{M}}\frac{\partial}{\partial y}+\bar{\omega}^{\mathrm{M}}\frac{\partial}{\partial p}\right)\bar{\theta}^{\mathrm{M}} = \bar{Q}^{\mathrm{M}}+D, \tag{28}$$

where $$D \equiv \partial(\bar{\theta}^{\mathrm{M}}-\bar{\theta}^{\mathrm{L}})/\partial t,$$

and where $\bar{\theta}^{\mathrm{L}}$ in the last expression is defined for material tubes C that coincide with the '$P\theta$ tubes' $\mathrm{C_M}$ at time t, so that $\overline{(\ )}^{\mathrm{M}} = \overline{(\ )}^{\mathrm{L}}$ everywhere at that instant. (Of course $\bar{\theta}^{\mathrm{M}}$ is simply equal to the value of θ for the tube $\mathrm{C_M}$, but $\bar{v}^{\mathrm{M}}$ and $\bar{\omega}^{\mathrm{M}}$ still involve non-trivial averaging.) The term D is written on the right because it is a *dissipative* term, entirely attributable to frictional and diabatic processes; were these zero, C and $\mathrm{C_M}$ would continue to coincide as time progressed and D would vanish.† D also represents an *eddy* process, because D would be zero if all tubes C and $\mathrm{C_M}$ were straight as in figure 2*b* for some time interval around time t. (It would be of some interest to express D explicitly in terms of disturbance-associated quantities, but that has not yet been done.) Similarly, the analogue of equation (19) will take the form

$$\partial\tilde{\rho}^{\mathrm{M}}/\partial t+\nabla\cdot(\tilde{\rho}^{\mathrm{M}}\bar{\boldsymbol{u}}^{\mathrm{M}}) = \mathscr{D}, \tag{29}$$

where $\tilde{\rho}^{\mathrm{M}}$ is given by replacing ξ in (17) by ξ^{m} (see § 3),‡ and $\mathscr{D}$ stands both for 'divergence' *and* 'dissipation' since, like D, it is evidently zero for conservative motion because $\xi^{\mathrm{m}} = \xi$ in that case. $\mathscr{D}$, also, is an eddy term.

It turns out that $\bar{Q}^{\mathrm{M}}$ has a simpler physical interpretation than that associated with (28). It suggests an alternative way of describing the mean meridional circulation, which relates it more directly to $\bar{Q}^{\mathrm{M}}$ and to the analogous quantity $\bar{Q}_P^{\mathrm{M}}$ corresponding to $X = P$ in (26). This is obtained by considering the rate at which mass crosses surfaces of constant P and θ (see, for example, Danielsen 1968, p. 518). For instance, the rate at which mass crosses a portion of the isentropic surface marked out by values of the potential vorticity lying between P and $P+\Delta P$ is

$$\Delta\mu(P,\theta) = \bar{Q}^{\mathrm{M}}\sigma(P,\theta)\,\Delta P, \tag{30}$$

where $\sigma(P,\theta)$ is the apparent 'density' of fluid in a meridional cross section in which P and θ are used as coordinates; that is to say $\sigma(P,\theta)$ is the total mass of the tube $\mathrm{C_M}(P,\theta)$ divided by its cross-sectional 'area' $\Delta P\,\Delta\theta$ in (P,θ) space. A derivation of (30) is given in the appendix. ($\Delta\mu$ is closely related to the quantity V defined in § 2.3, in the case of stationary waves and hence immobile isentropic surfaces.) The total mass flux across the finite portion of the isentropic surface marked out by values of P running between P_1 and P_2, say, is given by

$$\mu = \int_{P_1}^{P_2}\bar{Q}^{\mathrm{M}}(P,\theta)\,\sigma(P,\theta)\,\mathrm{d}P. \tag{31}$$

An exactly similar result involving $\bar{Q}_P^{\mathrm{M}}$ evidently holds for the rate at which mass crosses a surface of constant potential vorticity P.

It is interesting to consider for a moment what would be involved in calculating the radiative contribution $\overline{(Q_{(r)})}^{\mathrm{M}}$ to $\overline{(Q)}^{\mathrm{M}}$ to a level of approximation beyond that of Dunkerton (1978).

† Note incidentally that in the definition of D, $\partial/\partial t$ may be replaced by $\partial/\partial t+\bar{v}^{\mathrm{M}}\partial/\partial y+\bar{\omega}^{\mathrm{M}}\partial/\partial p$.
‡ To convert to pressure coordinates replace ζ by p^l, in the notation of equation (9), as well as z by p.

Since the averaging in (25) involves a weighted mean along the wavy, quasi-material tube C_M one would have to take into account the correlation between tropospheric cloudiness and patterns of troughs and ridges, as well as the fact that some tubes C_M will intersect the edge of the polar night. (Terms like $\frac{1}{2}\overline{\eta^2 (Q_{(r)})_{yy}}$ in equation (12) (but involving η^{m} rather than η) would be important if we were to try to describe the latter effect within the small-amplitude approximation.) Because of the notorious difficulty of the radiative-transfer calculation itself (Houghton 1978), the left-hand side of equation (30) may in actual practice be easier to estimate (by using tracer observations) than the right-hand side. Thus, relations like (30) and (31) may be more useful as constraints on estimates of turbulent mixing and radiative cooling in the lower stratosphere, than as predictors of tracer motion *from* such estimates.

The foregoing discussion has done no more than touch on some of the theoretical facts and difficulties that may have to be taken into account if 'the mean circulation' of the middle atmosphere is to be conceived of in a way more satisfactory than that afforded by the usual Eulerian-mean description. Enough has been said, perhaps, to show that even though Lagrangian-mean theory has furnished some important insights (lucidly brought out in the recent contributions of Dunkerton (1978), Matsuno (1979) and Plumb (1979)), we are still a long way from a quantitative theory of the mean circulation which truly embodies these insights.

I should like to thank D. G. Andrews, S. Death, T. Dunkerton, A. Eliassen, C.-P. Hsu, J. C. R. Hunt, J. D. Mahlman, T. Matsuno, H. Pearson, R. Quiroz, M. Uryu and S. V. Venkateswaran for stimulating discussions and helpful comments on this material. Dr M. Uryu kindly allowed me to quote his work on the Eady problem before publication.

Appendix. Some further relations involving $\overline{(\ )}^{\mathrm{M}}$

We first note that (25) can be rewritten as

$$\overline{(\ )}^{\mathrm{M}} \equiv \frac{1}{\sigma}\int_{C_M} (\ )\,\rho\alpha\,\mathrm{d}s, \tag{A 1}$$

where ρ is density, s is arc length along the tube $C_M(P, \theta)$, σ is defined by

$$\sigma(P, \theta) \equiv \int_{C_M} \rho\alpha\,\mathrm{d}s, \tag{A 2}$$

and α is defined by

$$\begin{aligned}\alpha^{-1} &\equiv |\nabla\theta \times \nabla P|,\\ &= |\nabla\theta|\,|\nabla_\theta P| = |\nabla P|\,|\nabla_P \theta|.\end{aligned} \tag{A 3}$$

Here ∇ is the three-dimensional gradient operator as before, ∇_θ denotes the gradient in a surface of constant θ, and ∇_P the gradient in a surface of constant P. In the notation of § 3, $\alpha\Delta P\Delta\theta$ is the cross-sectional area ΔA_M of the tube $C_M(P, \theta)$ and $\sigma\Delta P\Delta\theta$ its mass.

Now consider the rate $\delta\mu$ at which mass crosses an area element δA of an isentropic surface. That rate is $\rho\delta A$ times the component $u^\perp$ of fluid velocity normal to the isentropic surface, as seen in a frame of reference fixed in the isentropic surface. If the positive direction is taken towards higher θ, the material rate of change of potential temperature is just $u^\perp|\nabla\theta|$. But this is equal to Q by definition, and so

$$u^\perp = Q/|\nabla\theta|.$$

Substituting this result into our expression for $\delta\mu$, we get

$$\delta\mu = \rho Q\delta A/|\nabla\theta|. \tag{A 4}$$

Now the area element of the isentropic surface marked out by values P and $P+\Delta P$ of potential vorticity, and values s and $s+\mathrm{d}s$ of arc length along C_M, is

$$\delta A = \Delta P \mathrm{d}s/|\nabla_\theta P|;$$

so the mass flux across this area element is

$$\begin{aligned}\delta\mu &= \rho Q \Delta P \mathrm{d}s/|\nabla\theta|\;|\nabla_\theta P|\\ &= \Delta P Q \rho\alpha \mathrm{d}s,\end{aligned} \tag{A 5}$$

by (A 3). Equation (30) now follows, because (A 1) shows that the integral of (A 5) with respect to s is $\sigma\Delta P\bar{Q}^M$. Equation (31) is the result of carrying out a further integration, with respect to P.

It is perhaps worth noting for completeness the corresponding result when the integration is over any arbitrary area A of the isentropic surface, even though the result is hardly novel. The total rate at which mass crosses the area A is

$$\int Q\hat{\sigma}\mathrm{d}A, \tag{A 6}$$

where $\hat{\sigma} = \rho/|\nabla\theta|$ is the mass per unit area per unit increment in θ, that is to say $(\Delta\theta)^{-1}$ times the mass per unit area lying between the isentropic surfaces θ and $\theta+\Delta\theta$ at the point under consideration. This result remains useful even in cases where the (P, θ) coordinate system becomes ill-defined. (The expression (A 6) is $\overline{\int\hat{\sigma}\mathrm{d}A}$ times the modified Lagrangian mean of Q corresponding to the Eulerian spatial mean $\overline{(\;)}$ over a horizontal *surface*.)

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